GLOBAL EXISTENCE OF CLASSICAL SOLUTIONS FOR A NONLOCAL ONE DIMENSIONAL PARABOLIC FREE BOUNDARY PROBLEM

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Abstract. In this paper we study one dimensional parabolic free boundary value problem with a nonlocal (integro-differential) condition on the free boundary. We establish global existence–uniqueness of classical solutions assuming that the initial-boundary data are sufficiently smooth and satisfy some compatibility conditions. Our approach is based on analysis of an equivalent system of nonlinear integral equations.

Keywords: free boundary problem, parabolic equation, mixed type boundary conditions, system of nonlinear integral equations.

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1. Introduction

In this paper we consider the following free boundary value problem.
Problem P. Find $s(t) > 0$ and $u(x,t)$ such that

$$u_t = u_{xx} - \lambda u, \quad \lambda = const > 0, \quad 0 < x < s(t), \ \ t > 0; \tag{1.1}$$

$$s'(t) = \int_0^{s(t)} (u(x,t) - \tilde{\sigma})\, dx, \quad t > 0, \quad \tilde{\sigma} = const > 0; \tag{1.2}$$

$$u(0,t) = f(t), \quad f(t) > 0, \quad t \geq 0; \tag{1.3}$$

$$u(x,0) = \varphi(x), \quad \varphi(x) > 0, \ \ x \in [0,b], \ \ s(0) = b > 0, \quad \varphi(0) = f(0); \tag{1.4}$$

$$u_x(s(t),t) = 0, \quad t > 0. \tag{1.5}$$

This is one-dimensional free boundary problem with unknown boundary $x = s(t)$. Notice that (1.2) is a nonlocal condition on the free boundary, and (1.3)–(1.5) are mixed type boundary conditions for the parabolic equation (1.1).

The aim of this paper is to investigate the existence and uniqueness of classical solutions of Problem P. To this end we reduce the problem to a system of nonlinear integral equations and analyze its local solvability. The same approach has been used in many papers on the one-dimensional Stefan problem and its variations in order to prove existence-uniqueness results – see, for instance, [7, Ch. 8] and [12] and the bibliography given there. In

the context of tumor models, existence-uniqueness results for free boundary problems similar to Problem P are obtained in [8, Theorem 3.1] and [5, Theorem 2.1] by the same method. However, the presence of mixed type boundary conditions in Problem P brings to some additional difficulties, and as far as we know this case has not been studied yet.

Our main result is the following.

Theorem 1.1. *Suppose that*

$$f(t) \in C^1([0,\infty)), \quad f(t) > 0 \ \text{for} \ t \geq 0, \quad \varphi(x) \in C^2([0,b]), \tag{1.6}$$

$$\varphi(x) > 0 \ \text{for} \ x \in [0,b], \quad f(0) = \varphi(0), \quad f'(0) = \varphi''(0) - \lambda\varphi(0), \quad \varphi'(b) = 0.$$

Then there exists a unique pair of functions $u(x,t)$ and $s(t)$ such that

(i) $u(x,t)$ is defined, continuous and has continuous partial derivatives u_x, u_t, u_{xx} in the domain $\{(x,t): \ 0 \leq x \leq s(t), \ t \geq 0\}$;

(ii) $s(t) \in C^1([0,\infty))$;

(iii) (1.1)–(1.5) hold.

In order to prove this theorem we introduce an auxiliary free boundary value problem (see Problem $\tilde{P}$ in Section 3) and analyze local and global in time solvability of the resulting pair of free boundary value problems. In Lemma 3.1 it is shown that every solution of the main Problem P ((1.1)–(1.5)) generates a solution of the auxiliary problem and vice versa. Existence and uniqueness of local solutions of the auxiliary problem are proved by deriving and studying an equivalent system of nonlinear integral equations (see Lemma 4.2 and Lemma 5.1). In Lemma 6.1 we obtain a priori estimates for the local solutions of the auxiliary problem by applying an appropriate maximum principle (see Lemma 2.1 and Lemma 2.2) to the solutions of the main problem. Finally, we prove existence of global solutions for both the main and the auxiliary problems by using the corresponding a priori estimates obtained in Lemma 6.1. Some of these results are announced without proofs in [15].

2. Preliminary results

Throughout the paper we assume that the functions f and φ satisfy the conditions (1.6).

Definition 1. We say that a pair of functions $(u(x,t), s(t))$ is a solution of Problem P for $t \in [0,T), \ T \leq \infty$, if

(i) $u(x,t)$ is defined, continuous and has continuous partial derivatives u_x, u_t, u_{xx} in the domain $D_T = \{(x,t): \ 0 \leq x \leq s(t), \ 0 \leq t < T\}$;

(ii) the equation (1.1) is satisfied for $t < T$;

(iii) $s(t) \in C^1([0,T))$;

(iv) the conditions (1.2)–(1.5) hold for $t \in [0,T)$.

Lemma 2.1. *(Maximum Principle) Let $\lambda = const > 0, \ s(t) \in C^1([0,T])$, and let $u(x,t)$ be defined and continuous in the domain $\overline{D}_T = \{(x,t): \ 0 \leq$*

$x \le s(t)$, $0 \le t \le T\}$, have continuous partial derivatives u_x, u_{xx} for $0 < x \le s(t)$, $0 < t \le T$, and have a continuous partial derivative u_t for $0 < x < s(t)$, $0 < t < T$. Suppose that

$$-\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} - \lambda u \ge 0 \quad \text{for } 0 < x < s(t),\ 0 < t < T, \tag{2.1}$$

and

$$\frac{\partial u}{\partial x}(s(t), t) \le 0 \quad \text{for } 0 \le t \le T. \tag{2.2}$$

If $M := \max\{u(x,t) : (x,t) \in \overline{D}_T\} > 0$, then $u(x,t)$ attains its maximum only on the union of the segments $\{(0,t) : t \in [0,T]\}$ and $\{(x,0) : x \in [0, s(0)]\}$.

Proof. To the contrary, assume that $u(x,t)$ attains its maximum $M > 0$ at a point (x_0, t_0) with $x_0 > 0$, $t_0 > 0$.

(a) Suppose $x_0 < s(t_0)$. Obviously, $u_x(x_0, t_0) = 0$. On the other hand, in view of (2.1), we have

$$0 \le \frac{1}{h}\left(u(x_0,t_0) - u(x_0, t_0 - h)\right) = u_t(x_0, t_h) \le u_{xx}(x_0, t_h) - \lambda u(x_0, t_h),$$

with $0 < h < t_0$ and $t_0 - h < t_h < t_0$. Therefore, letting $h \to 0$, we get $u_{xx}(x_0, t_0) \ge \lambda M > 0$. But then the function of one variable $u(x, t_0)$ has a strict local minimum at $x = x_0$, which is impossible.

(b) Assume that $x_0 = s(t_0)$. Since $s'(t_0)$ exists, there is a unit vector $\vec{\ell} = (\alpha, \beta)$ with $\alpha > 0$ and $\beta > 0$ such that the segment $\{(s(t_0) - \alpha h, t_0 - \beta h),\ h \in (0, \varepsilon)\}$ is in the interior of D_T for sufficiently small $\varepsilon > 0$. Then

$$0 \le \frac{1}{h}\left(u(s(t_0), t_0) - u(s(t_0) - \alpha h, t_0 - \beta h)\right) = \frac{\partial u}{\partial \vec{\ell}}(x_h, t_h),$$

where $x_h = s(t_0) - \alpha\theta h,\ t_h = t_0 - \beta\theta h,\ \theta = \theta(h) \in (0,1)$. Therefore, by (2.1) it follows

$$0 \le \alpha u_x(x_h, t_h) + \beta u_t(x_h, t_h) \le \alpha u_x(x_h, t_h) + \beta(u_{xx}(x_h, t_h) - \lambda u(x_h, t_h)).$$

Passing to a limit as $h \to 0$ we get

$$0 \le \alpha u_x(s(t_0), t_0) + \beta(u_{xx}(s(t_0), t_0) - \lambda u(s(t_0), t_0)).$$

By (2.2) we have $u_x(s(t_0), t_0) \le 0$. Therefore, the latter inequality yields $u_{xx}(s(t_0), t_0) \ge \lambda M > 0$, which implies that $u_x(s(t_0) - h, t_0) < 0$ for all sufficiently small $h > 0$, say $h \in (0, \delta)$. But then it follows that $u(s(t_0) - h, t_0) > u(s(t_0), t_0)) = M$ for $h \in (0, \delta)$, which is impossible. This completes the proof. □

In view of (1.2), the Maximum Principle yields immediately the following a priori estimates for $u(x,t)$ and $s(t)$.

Lemma 2.2. *If a pair of functions $(u(x,t), s(t))$ is a solution of Problem P for $0 \le t < T < \infty$, then*

$$0 \le u(x,t) \le C_T, \quad 0 \le x \le s(t),\ \ 0 \le t < T, \tag{2.3}$$

$$(2.4)\qquad -\tilde{\sigma}s(t) \le s'(t) \le (C_T - \tilde{\sigma})s(t), \quad be^{-\tilde{\sigma}t} \le s(t) \le be^{(C_T-\tilde{\sigma})t},$$

where $b = s(0),\ C_T = \max\left(\sup_{[0,T)} f(t),\ \sup_{[0,b]} \varphi(x)\right).$

3. Auxiliary free boundary problem

Next we consider the following *auxiliary free boundary value problem.*

Problem $\tilde{\mathbf{P}}$. Find $s(t) > 0$ and $\tilde{u}(x,t)$ such that

$$(3.1)\qquad \tilde{u}_t = \tilde{u}_{xx}, \quad 0 < x < s(t),\ \ t > 0;$$

$$(3.2)\qquad s'(t) = (f(t) - \tilde{\sigma})s(t) + e^{-\lambda t}\int_0^{s(t)} \left(\int_0^x \tilde{u}(\xi,t)d\xi\right) dx, \quad t > 0;$$

$$(3.3)\qquad \tilde{u}_x(0,t) = \tilde{f}(t), \quad \tilde{f}(t) \in C([0,\infty));$$

$$(3.4)\quad \tilde{u}(x,0) = \tilde{\varphi}(x),\ \tilde{\varphi}(x) \in C^1([0,b]),\ s(0) = b,\ \tilde{\varphi}'(0) = \tilde{f}(0),\ \tilde{\varphi}(b) = 0;$$

$$(3.5)\qquad \tilde{u}(s(t),t) = 0, \quad t > 0.$$

Definition 2. We say that a pair of functions $\tilde{u}(x,t)$ and $s(t)$ is a solution of Problem $\tilde{P}$ for $t \in [0,T),\ T \le \infty$, if

(i) $\tilde{u}(x,t)$ is defined and continuous in the domain $D_T = \{(x,t) :\ 0 \le x \le s(t),\ 0 \le t < T\}$, has continuous partial derivative $\tilde{u}_x$ in D_T, and has continuous partial derivatives $\tilde{u}_t,\ \tilde{u}_{xx}$ for $0 < x < s(t),\ 0 < t < T$;

(ii) the equation (3.1) is satisfied for $t \in (0,T)$;

(iii) $s(t) \in C^1([0,T))$;

(iv) the conditions (3.2)–(3.5) hold for $t \in [0,T)$.

The next lemma gives the relation between Problem P and Problem $\tilde{P}$.

Lemma 3.1. *Let $f(t)$ and $\varphi(x)$ satisfy (1.6), $\lambda = const > 0$, and let*

$$(3.6)\qquad \tilde{f}(t) = \frac{d}{dt}\left(e^{\lambda t} f(t)\right), \quad \tilde{\varphi}(x) = \varphi'(x).$$

(a) If a pair of functions $u(x,t)$ and $s(t)$ is a solution of Problem P for $t \in [0,T)$, then the pair of functions $\tilde{u}(x,t) = e^{\lambda t}u_x(x,t)$ and $s(t)$ is a solution of Problem $\tilde{P}$ for $t \in [0,T)$.

(b) If a pair of functions $\tilde{u}(x,t)$ and $s(t)$ is a solution of Problem $\tilde{P}$ for $t \in [0,T)$, then the pair of functions $(u(x,t), s(t))$ with

$$(3.7)\qquad u(x,t) = f(t) + e^{-\lambda t}\int_0^x \tilde{u}(\xi,t)d\xi$$

is a solution of Problem P for $t \in [0,T)$.

Proof. (a) Notice that $u(x,t)$ is a C^∞-function in the interior of the domain D_T due to general smoothness theorems (see [7, Ch.3, Thm. 11], and Corollary 2 there). Therefore, the function $\tilde{u}(x,t)$ has continuous partial derivatives $\tilde{u}_t$, $\tilde{u}_{xx}$ for $0 < x < s(t)$, $0 < t < T$, and it satisfies the equation $\tilde{u}_t = \tilde{u}_{xx}$ in that domain.

Letting $x \to 0$ in the equation (1.1), we obtain

$$u_t(0,t) = f'(t) = u_{xx}(0,t) - \lambda f(t).$$

Thus, $\tilde{u}_x(0,t) = u_{xx}(0,t)e^{\lambda t} = [f'(t) + \lambda f(t)]e^{\lambda t}$, i.e., (3.3) holds with $\tilde{f}(t) = \frac{d}{dt}\left(f(t)e^{\lambda t}\right)$. Now one can readily verify that the pair of functions $\tilde{u}(x,t) = e^{\lambda t}u_x(x,t)$ and $s(t)$ is a solution of Problem $\tilde{P}$ for $t \in [0,T)$.

(b) We check first that the function $u(x,t)$ given in (3.7) satisfies the equation (1.1). By (3.7), we have

$$u_{xx}(x,t) = e^{-\lambda t}\tilde{u}_x(x,t).$$

In order to find and justify a formula for u_t we set

$$u_n(x,t) = f(t) + e^{-\lambda t}\int_{1/n}^{x} \tilde{u}(\xi,t)d\xi, \quad n = 1,2,\dots.$$

Then $u_n(x,t) \to u(x,t)$ as $n \to \infty$ for $(x,t) \in D_T$. In view of (3.1), we have

$$\frac{\partial}{\partial t}(u_n(x,t)) = f'(t) - \lambda e^{-\lambda t}\int_{1/n}^{x} \tilde{u}(\xi,t)d\xi + e^{-\lambda t}\int_{1/n}^{x} \tilde{u}_{\xi\xi}(\xi,t)d\xi.$$

Since $\int_{1/n}^{x} \tilde{u}_{\xi\xi}(\xi,t)d\xi = \tilde{u}_x(x,t) - \tilde{u}_x(1/n,t)$, we get as $n \to \infty$

$$\frac{\partial}{\partial t}(u_n(x,t)) \to f'(t) - \lambda e^{-\lambda t}\int_{0}^{x} \tilde{u}(\xi,t)d\xi + e^{-\lambda t}(\tilde{u}_x(x,t) - \tilde{u}_x(0,t))$$

uniformly on any compact subinterval of $(0,T)$. Therefore, $u_t(x,t)$ exists, and using (3.7) and (3.3) we obtain

$$u_t(x,t) = f'(t) - \lambda(u(x,t) - f(t)) + e^{-\lambda t}(\tilde{u}_x(x,t) - \tilde{f}(t)).$$

Since $\tilde{f}(t) = \frac{d}{dt}\left(f(t)e^{\lambda t}\right)$ it follows that $u(x,t)$ satisfies the equation (1.1). Now one can easily see that the pair of functions $(u(x,t), s(t))$ is a solution of Problem P for $t \in [0,T)$. □

In view of Lemma 3.1, Theorem 1.1 will be proved if we show that the following statement holds.

Theorem 3.2. *Suppose that*

$$f(t) \in C^1([0,\infty)), \quad f(t) > 0 \ \textit{for}\ t \geq 0, \quad \tilde{f}(t) \in C([0,\infty)), \tag{3.8}$$

$$\tilde{\varphi}(x) \in C^1([0,b]), \qquad \tilde{f}(0) = \tilde{\varphi}'(0), \quad \tilde{\varphi}(b) = 0.$$

Then Problem $\tilde{P}$ has a unique solution for $0 \leq t < \infty$.

4. System of integral equations

In this section Problem $\tilde{P}$ is transformed to an equivalent problem of solving a system of nonlinear integral equations. We begin with some preliminaries.

Consider the function

$$K(x,t;\xi,\tau)=\frac{1}{2\sqrt{\pi}\sqrt{t-\tau}}\exp\left(-\frac{(x-\xi)^2}{4(t-\tau)}\right),\quad \tau<t. \tag{4.1}$$

We shall make use of the following elementary inequalities:

$$\int_0^t |K_x(x,t;\xi,\tau)|\,d\tau=\frac{1}{\sqrt{\pi}}\int_{\frac{|x-\xi|}{2\sqrt{t}}}^{\infty} e^{-z^2}dz\le\frac{1}{2} \tag{4.2}$$

(by performing the change of variable $z=\frac{|x-\xi|}{2\sqrt{t-\tau}}$);

$$\int_0^b K(x,t;\xi,0)d\xi=\frac{1}{\sqrt{\pi}}\int_{\frac{-x}{2\sqrt{t}}}^{\frac{b-x}{2\sqrt{t}}} e^{-z^2}dz\le 1 \tag{4.3}$$

(by using the change of variable $z=\frac{\xi-x}{2\sqrt{t}}$);

$$z^a e^{-\gamma z}\le\left(\frac{a}{e\gamma}\right)^a \quad \text{if}\quad z>0,\ a>0,\ \gamma>0. \tag{4.4}$$

The next statement is a slight modification of Lemma 1 in [7, Ch.8]), and its proof is the same.

Lemma 4.1. *If $s(t)\in C^1([0,T]),\ g(t)\in C([0,T])$ and $0<t_0<T$, then*

$$\lim_{(x,t)\to(s(t_0),t_0)}\int_0^t K_x(x,t;s(\tau),\tau)g(\tau)d\tau=\frac{g(t_0)}{2}+\int_0^t K_x(s(t_0),t_0;s(\tau),\tau)g(\tau)d\tau,$$

where in the limit we consider only points (x,t) with $x<s(t)$.

Next we derive a system of integral equations related to Problem $\tilde{P}$. Let $N(x,t;\xi,\tau)$ be the Neumann function for the half-plane $x>0$, i.e.,

$$N(x,t;\xi,\tau)=K(x,t;\xi,\tau)+K(-x,t;\xi,\tau).$$

Suppose that the pair of functions $(\tilde{u}(x,t),s(t))$ is a solution of Problem $\tilde{P}$. For $t>0$, we integrate the identity

$$\frac{\partial}{\partial\xi}\left(N\frac{\partial\tilde{u}}{\partial\xi}-\frac{\partial N}{\partial\xi}\tilde{u}\right)=\frac{\partial}{\partial\tau}(N\tilde{u}),\quad \tilde{u}=\tilde{u}(\xi,\tau), \tag{4.5}$$

over the domain $\varepsilon\le\tau\le t-\varepsilon,\quad \delta\le\xi\le s(\tau)-\delta,\quad \varepsilon=const>0,\ \delta=const>0$, and pass to limits, first as $\delta\to 0$, and then as $\varepsilon\to 0$. Since $N_\xi(x,t;0,\tau)=0$ and

$$\lim_{\varepsilon\to 0}\int_0^{s(t-\varepsilon)} N(x,t;\xi,t-\varepsilon)\tilde{u}(\xi,t-\varepsilon)d\xi=\tilde{u}(x,t),$$

it follows, in view of (3.3) and (3.4), that

$$\tilde{u}(x,t) = \sum_{\nu=1}^{5} J_\nu(x,t), \tag{4.6}$$

where

$$J_1(x,t) = \int_0^t N(x,t;s(\tau),\tau)v(\tau)d\tau \qquad \text{with} \quad v(t) := \tilde{u}_x(s(t),t), \tag{4.7}$$

$$J_2(x,t) = -\int_0^t N(x,t;0,\tau)\tilde{f}(\tau)d\tau, \quad J_3(x,t) = \int_0^b N(x,t;\xi,0)\tilde{\varphi}(\xi)d\xi$$

and

$$J_4(x,t) = -\int_0^t N_\xi(x,t;s(\tau),\tau)\tilde{u}(s(\tau),\tau)d\tau, \tag{4.8}$$

$$J_5(x,t) = \int_0^t s'(\tau)N(x,t;s(\tau),\tau)\tilde{u}(s(\tau),\tau)d\tau.$$

The condition (3.5) implies $J_4(x,t) = 0,\ J_5(x,t) = 0$. Thus, the following integral representation holds:

$$\tilde{u}(x,t) = J_1(x,t) + J_2(x,t) + J_3(x,t). \tag{4.9}$$

Next, in order to obtain an integral equation for $v(t) = \tilde{u}_x(s(t),t)$, we differentiate (4.9) with respect to x and pass to a limit as $x \to s(t) - 0$ in the resulting identity. In view of Lemma 4.1, it follows that

$$\lim_{x\to s(t)-0} \frac{\partial J_1}{\partial x}(x,t) = \int_0^t N_x(s(t),t;s(\tau),\tau)v(\tau)d\tau + \frac{v(t)}{2}.$$

It is easy to see that

$$\lim_{x\to s(t)-0} \frac{\partial J_2}{\partial x}(x,t) = -\int_0^t N_x(s(t),t;0,\tau)\tilde{f}(\tau)d\tau.$$

Now, consider the Green function for the half–plane $x > 0$

$$G(x,t;\xi,\tau) = K(x,t;\xi,\tau) - K(-x,t;\xi,\tau).$$

Since $N_x = -G_\xi$, an integration by parts leads to

$$\frac{\partial J_3}{\partial x}(x,t) = -\int_0^b G_\xi(x,t;\xi,0)\tilde{\varphi}(\xi)d\xi = \int_0^b G(x,t;\xi,0)\tilde{\varphi}'(\xi)d\xi \tag{4.10}$$

because $G(x,t;0,0) = 0$ and $\tilde{\varphi}(b) = 0$. Therefore, it follows that

$$\lim_{x\to s(t)-0} \frac{\partial J_3}{\partial x}(x,t) = \int_0^b G(s(t),t;\xi,0)\tilde{\varphi}'(\xi)d\xi.$$

Hence, for $t > 0$ the function $v(t)$ satisfies the integral equation

$$v(t) = 2\int_0^t N_x(s(t),t;s(\tau),\tau)v(\tau)d\tau \tag{4.11}$$

$$-2\int_0^t N_x(s(t),t;0,\tau)\tilde{f}(\tau)d\tau + 2\int_0^b G(s(t),t;\xi,0)\tilde{\varphi}'(\xi)d\xi.$$

On the other hand, from (3.2) and (4.9) it follows

(4.12)

$$s'(t) = (f(t)e^{-\lambda t} - \tilde{\sigma})s(t) + e^{-\lambda t}\left(\int_0^{s(t)}\int_0^x\int_0^t N(\xi,t;s(\tau),\tau)v(\tau)d\tau d\xi dx\right.$$

$$\left. - \int_0^{s(t)}\int_0^x\int_0^t N(\xi,t;0,\tau)\tilde{f}(\tau)d\tau d\xi dx + \int_0^{s(t)}\int_0^x\int_0^b N(\xi,t;\eta,0)\tilde{\varphi}(\eta)d\eta d\xi dx\right).$$

The system of nonlinear integral equations (4.11) and (4.12) considered with $s(t) = b + \int_0^t s'(\tau)d\tau$ is equivalent to Problem $\tilde{P}$, i.e., the following statement holds.

Lemma 4.2. *Problem $\tilde{P}$ for $t < T$ is equivalent to the problem of finding a pair of continuous functions $(v(t),\ s'(t))$ on $[0,T)$ which satisfies for $t > 0$ the system of nonlinear integral equations (4.11) and (4.12) considered with $s(t) = b + \int_0^t s'(\tau)d\tau$.*

Proof. We have already proved that if a pair $(\tilde{u}(x,t), s(t))$ is a solution of Problem $\tilde{P}$ for $t < T$, then the pair of continuous functions $v(t) = \tilde{u}_x(s(t),t)$ and $s'(t),\ t \in [0,T)$, satisfies for $t > 0$ the system of nonlinear integral equations (4.11) and (4.12) considered with $s(t) = b + \int_0^t s'(\tau)d\tau$.

Conversely, suppose that a pair of continuous functions $v(t)$ and $s'(t),\ t \in [0,T)$, satisfies for $t > 0$ the system of integral equations (4.11) and (4.12). Set

$$\tilde{u}(x,t) = \begin{cases} \sum_{\nu=1}^3 J_\nu(x,t) & \text{for } 0 \le x \le s(t),\ 0 < t < T, \\ \tilde{\varphi}(x) & \text{for } 0 \le x \le b, \quad t = 0, \end{cases} \tag{4.13}$$

where $J_\nu(x,t),\ \nu = 1,2,3$ are given by (4.7) and $s(t) = b + \int_0^t s'(\tau)d\tau$. We shall prove that the pair of functions $(\tilde{u}(x,t), s(t))$ form a solution of Problem $\tilde{P}$ for $t < T$.

First we show that the function $\tilde{u}(x,t)$ is continuous in the domain D_T. Indeed, since the integrands in $J_1(x,t)$ and $J_2(x,t)$ are dominated by a multiple of $(t-\tau)^{-1/2}$, we have

$$\lim_{(x,t)\to(x_0,0)} J_\nu(x,t) = 0, \quad \nu = 1,2, \quad x_0 \in [0,b].$$

So, it remains to show that $J_3(x,t) \to \tilde{\varphi}(x_0)$ as $(x,t) \to (x_0,0),\ x_0 \in [0,b]$. Performing the change of variable $z = (\xi \pm x)/2\sqrt{t}$, we obtain $J_3(x,t) = J_3^1(x,t) + J_3^2(x,t)$, where

$$J_3^1(x,t) = \frac{1}{\sqrt{\pi}}\int_{-\frac{x}{2\sqrt{t}}}^{\frac{b-x}{2\sqrt{t}}} e^{-z^2}\tilde{\varphi}(x+2z\sqrt{t})dz, \quad J_3^2(x,t) = \frac{1}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{t}}}^{\frac{b+x}{2\sqrt{t}}} e^{-z^2}\tilde{\varphi}(-x+2z\sqrt{t})dz.$$

Now, for every $x_0 \in (0,b)$, it follows that

$$J_3^1(x,t) \to \tilde{\varphi}(x_0) \quad \text{and} \quad J_3^2(x,t) \to 0 \quad \text{as } (x,t) \to (x_0, 0)$$

because $-x/2\sqrt{t} \to -\infty$ and $(b \pm x)/2\sqrt{t} \to +\infty$.

The corner points $(0,0)$ and $(b,0)$ need a special consideration. If $(x,t) \to (b,0)$, then $J_3^2(x,t) \to 0$ by the same argument. Since $\tilde{\varphi}(b) = 0$, it follows that $J_3^1(x,t) \to 0$ as well, so $J_3(x,t) \to 0 = \tilde{\varphi}(b)$ as $(x,t) \to (b,0)$.

In order to show that $J_3(x,t) \to \tilde{\varphi}(0)$ as $(x,t) \to (0,0)$ we shall prove that $J_3(x_n,t_n) \to \tilde{\varphi}(0)$ for every sequence $(x_n,t_n) \to (0,0)$. Since $J_3^1(x,t)$ and $J_3^2(x,t)$ are bounded, the sequence $\{J_3(x_n,t_n)\}$ is bounded. Therefore, it is enough to show that every convergent subsequence of the form $\{J_3(x_{n_k},t_{n_k})\}$ has a limit equal to $\tilde{\varphi}(0)$. We may assume that $x_{n_k}/2\sqrt{t_{n_k}} \to a \in [0,\infty]$ (otherwise we may pass to a subsequence of (n_k)). Then it follows that

$$J_3(x_{n_k},t_{n_k}) \to \frac{1}{\sqrt{\pi}} \int_{-a}^{\infty} e^{-z^2} \tilde{\varphi}(0)dz + \frac{1}{\sqrt{\pi}} \int_{a}^{\infty} e^{-z^2} \tilde{\varphi}(0)dz = \tilde{\varphi}(0).$$

Thus, $J_3(x,t) \to \tilde{\varphi}(0)$ as $(x,t) \to (0,0)$, which completes the proof of continuity of $\tilde{u}(x,t)$ in the domain D_T.

It is easy to see that each of the integrals $J_\nu(x,t)$ is a C^∞–function in the domain $0 < x < s(t),\ 0 < t < T$, and satisfies the heat equation there. Thus, $\tilde{u}(x,t)$ satisfies (3.1) as well.

The functions $v(t)$ and $s'(t)$ are defined and continuous on $[0,T)$ and satisfy the integral equations (4.11) and (4.12) for $t \in (0,T)$. Consider the limit of the right-hand side of (4.11) as $t \to 0$. It is easy to see that the first two integrals there converge to zero. With $z = (\xi \pm s(t))/2\sqrt{t}$, the integral $\int_0^b G(s(t),t;\xi,0)\tilde{\varphi}'(\xi)d\xi$ is equal to

$$\frac{1}{\sqrt{\pi}} \int_{-\frac{s(t)}{2\sqrt{t}}}^{\frac{b-s(t)}{2\sqrt{t}}} e^{-z^2} \tilde{\varphi}'(s(t)+2z\sqrt{t})dz - \frac{1}{\sqrt{\pi}} \int_{\frac{s(t)}{2\sqrt{t}}}^{\frac{b+s(t)}{2\sqrt{t}}} e^{-z^2} \tilde{\varphi}'(-s(t)+2z\sqrt{t})dz.$$

As $t \to 0$, the first integral in the above expression tends to $\tilde{\varphi}(b)/2$, while the second one tends to zero. Therefore, by passing to limit as $t \to 0$ in (4.11) we obtain

$$v(0) = \tilde{\varphi}'(b). \tag{4.14}$$

Next we prove that $\tilde{u}_x(x,t)$ extends as a continuous function on D_T. In view of (4.7) and (4.9)–(4.10), we have

$$\tilde{u}_x(x,t) = I_1(x,t) + I_2(x,t) + I_3(x,t), \quad (x,t) \in D_T^\circ, \tag{4.15}$$

where $D_T^\circ = \{(x,t) :\ 0 < x < s(t),\ 0 < t < T\}$, and

$$I_1(x,t) = \int_0^t N_x(x,t;s(\tau),\tau)v(\tau)d\tau, \tag{4.16}$$

$$I_2(x,t) = -\int_0^t N_x(x,t;0,\tau)\tilde{f}(\tau)d\tau, \quad I_3(x,t) = \int_0^b G(x,t;\xi,0)\tilde{\varphi}'(\xi)d\xi.$$

We shall prove that

$$(i)\quad \tilde{u}_x(x,t)\to \tilde{f}(t_0)\quad \text{as}\quad (x,t)\to(0,t_0),\ \ (x,t)\in D_T^\circ,\ \ 0<t_0<T;$$

$$(ii)\quad \tilde{u}_x(x,t)\to v(t_0)\quad \text{as}\quad (x,t)\to(s(t_0),t_0),\ \ (x,t)\in D_T^\circ,\ \ 0<t_0<T;$$

$$(iii)\quad \tilde{u}_x(x,t)\to \tilde{\varphi}'(x_0)\quad \text{as}\quad (x,t)\to(x_0,0),\ \ (x,t)\in D_T^\circ,\ \ 0\le x_0\le b.$$

Since the functions $\tilde{f}(t),\ \tilde{\varphi}'(x),\ v(t)$ are continuous and $\tilde{f}(0)=\tilde{\varphi}'(0),\ \tilde{\varphi}'(b)=v(0)$ (see (3.4) and (4.14)), the conditions $(i)-(iii)$ guarantee that $\tilde{u}_x$ extends as a continuous function on D_T.

First we prove (i). Taking into account that $N_x(0,t,\xi,\tau)=0$ and $G(0,t,\xi,\tau)=0$, it is easy to see that $I_\nu(x,t)\to(0,t_0)$ for $\nu=1,3$ as $x\to+0,\ t\to t_0\in(0,T)$. With the change of variable $z=\frac{x}{2\sqrt{t-\tau}}$ the integral $I_2(x,t)$ becomes

$$I_2(x,t)=-\int_0^t N_x(x,t;0,\tau)\tilde{f}(\tau)d\tau=\frac{2}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{t}}}^{\infty} e^{-z^2}\tilde{f}\left(t-\frac{x^2}{4z^2}\right)dz.$$

As $x\to+0,\ t\to t_0\in(0,T)$ the latter integral tends to $\tilde{f}(t_0)$. Thus (i) holds.

Next we prove (ii). One can easily see that

$$I_\nu(x,t)\to I_\nu(s(t_0),t_0)\quad \text{as}\quad (x,t)\to(s(t_0),t_0),\quad \nu=2,3.$$

From Lemma 4.1 and (4.16) it follows that

$$\lim_{(x,t)\to(s(t_0),t_0)} I_1(x,t)=\frac{1}{2}v(t_0)+I_1(s(t_0),t_0).$$

Therefore, by (4.11), we obtain

$$\tilde{u}_x(x,t)\to\frac{1}{2}v(t_0)+\sum_{\nu=1}^{3}I_\nu(s(t_0),t_0)=v(t_0)\quad \text{as}\quad (x,t)\to(s(t_0),t_0),$$

i.e., (ii) holds.

It is easy to verify (iii) for $x_0\in(0,b)$. However, it is much more complicated to prove (iii) for $x_0=0$ or $x_0=b$.

Next we show that (iii) holds for $x_0=b$, i.e.,

$$\tilde{u}_x(x,t)\to\tilde{\varphi}'(b)\quad \text{as}\quad (x,t)\to(b,0),\ \ (x,t)\in D_T^\circ. \tag{4.17}$$

Let $\{(x_n,t_n)\}$ be an arbitrary sequence such that $(x_n,t_n)\to(b,0),\ (x_n,t_n)\in D_T^\circ$. In order to prove that $\tilde{u}_x(x_n,t_n)\to\tilde{\varphi}'(b)$ it is enough to show that for every subsequence $\{(x_{n_k},t_{n_k})\}$ there is a sub-subsequence $\{(x_{n_{k_m}},t_{n_{k_m}})\}$ (which we denote for convenience by $\{(x_m,t_m)\}$) such that

$$\tilde{u}_x(x_m,t_m)\to\tilde{\varphi}'(b)\quad \text{as}\ \ m\to\infty.$$

We may assume without loss of generality (otherwise one may pass to an appropriate subsequence) that

$$\frac{b-x_m}{2\sqrt{t_m}}\to\alpha\in[0,\infty]\quad \text{as}\ \ m\to\infty. \tag{4.18}$$

(Notice that $x_m < s(t_m)$ and

$$\frac{b - s(t_m)}{2\sqrt{t_m}} = \frac{s(0) - s(t_m)}{2\sqrt{t_m}} \to 0 \quad \text{as } m \to \infty;$$

therefore, every cluster point α in (4.18) is nonnegative.)

In view of (4.15), in order to evaluate $\lim_{m\to\infty} \tilde{u}_x(x_m, t_m)$ one needs to find $\lim_{m\to\infty} I_\nu(x_m, t_m)$, $\nu = 1, 2, 3$. First, consider the case $\nu = 1$. We have $I_1(x, t) = I_{1,1}(x, t) + I_{1,2}(x, t)$, where

$$I_{1,1}(x,t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{s(\tau) - x}{4(t-\tau)^{3/2}} e^{-\frac{(x-s(\tau))^2}{4(t-\tau)}} v(\tau) d\tau, \tag{4.19}$$

$$I_{1,2}(x,t) = -\frac{1}{\sqrt{\pi}} \int_0^t \frac{s(\tau) + x}{4(t-\tau)^{3/2}} e^{-\frac{(x+s(\tau))^2}{4(t-\tau)}} v(\tau) d\tau. \tag{4.20}$$

One can easily see that $I_{1,2}(x, t) \to 0$ as $(x, t) \to (b, 0)$ because $x + s(\tau) > b > 0$ for (x, τ) close to $(b, 0)$. On the other hand,

$$I_{1,1}(x_m, t_m) = I^1_{1,1}(x_m, t_m) + I^2_{1,1}(x_m, t_m) + I^3_{1,1}(x_m, t_m),$$

where

$$I^1_{1,1}(x_m, t_m) = \frac{1}{\sqrt{\pi}} \int_0^{t_m} \frac{s(\tau) - s(t_m)}{4(t_m-\tau)^{3/2}} e^{-\frac{[x_m-s(\tau)]^2}{4(t_m-\tau)}} v(\tau) d\tau,$$

$$I^2_{1,1}(x_m, t_m) = \frac{1}{\sqrt{\pi}} \int_0^{t_m} \frac{s(t_m) - x_m}{4(t_m-\tau)^{3/2}} \left[e^{-\frac{[x_m-s(\tau)]^2}{4(t_m-\tau)}} - e^{-\frac{[x_m-s(t_m)]^2}{4(t_m-\tau)}} \right] v(\tau) d\tau,$$

$$I^3_{1,1}(x_m, t_m) = \frac{1}{\sqrt{\pi}} \int_0^{t_m} \frac{s(t_m) - x_m}{4(t_m-\tau)^{3/2}} e^{-\frac{[x_m-s(t_m)]^2}{4(t_m-\tau)}} v(\tau) d\tau.$$

Since $s'(t)$ is continuous, the Mean Value Theorem implies that $|s(t_m) - s(\tau)| \le const \cdot |t_m - \tau|$. Therefore, the absolute value of the integrand of $I^1_{1,1}(x_m, t_m)$ does not exceed $C/\sqrt{t_m - \tau}$, which leads to $I^1_{1,1}(x_m, t_m) \le C\sqrt{t_m} \to 0$ as $m \to \infty$.

Changing the variable in $I^3_{1,1}$ by $\frac{s(t_m)-x_m}{2\sqrt{t_m-\tau}} = z$, we obtain

$$I^3_{1,1}(x_m, t_m) = \frac{1}{\sqrt{\pi}} \int_{\frac{s(t_m)-x_m}{2\sqrt{t_m}}}^{\infty} e^{-z^2} v\left(t_m - \frac{(s(t_m) - x_m)^2}{4z^2}\right) dz \to v(0) \cdot \frac{1}{\sqrt{\pi}} \int_\alpha^\infty e^{-z^2} dz.$$

The expression in the square brackets in the integral $I^2_{1,1}$ can be written as $\exp\left(-\frac{(x_m-s(t_m))^2}{4(t_m-\tau)}\right)\left(e^{g_m(\tau)} - 1\right)$, where

$$g_m(\tau) = \frac{s(t_m) - s(\tau)}{4(t_m - \tau)} \cdot (s(t_m) + s(\tau) - 2x_m) \to 0 \quad \text{as } m \to \infty$$

uniformly for $\tau \in [0, t_m]$. Therefore, the same change of variable as in $I^3_{1,1}$ shows that $I^2_{1,1}(x_m, t_m) \to 0$ as $m \to \infty$.

Hence, we obtain

$$I_1(x_m, t_m) \to v(0) \cdot \frac{1}{\sqrt{\pi}} \int_\alpha^\infty e^{-z^2} dz \quad \text{as } m \to \infty. \tag{4.21}$$

It is easy to see that

$$I_2(x_m,t_m)\to 0 \quad \text{as} \;\; m\to\infty. \tag{4.22}$$

Next we evaluate $\lim_{m\to\infty} I_3(x_m,t_m)$. Since $G(x,t;\xi,\tau)=K(x,t;\xi,\tau)-K(-x,t;\xi,\tau)$, performing the change of variable $z=(\xi\mp x)/2\sqrt{t}$ we obtain that $I_3(x,t)=I_{3,1}(x,t)+I_{3,2}(x,t)$, where

$$I_{3,1}=\frac{1}{\sqrt{\pi}}\int_{-\frac{x}{2\sqrt{t}}}^{\frac{b-x}{2\sqrt{t}}} e^{-z^2}\tilde{\varphi}'(x+2z\sqrt{t})dz, \quad I_{3,2}=\frac{1}{\sqrt{\pi}}\int_{\frac{x}{2\sqrt{t}}}^{\frac{b+x}{2\sqrt{t}}} e^{-z^2}\tilde{\varphi}'(-x+2z\sqrt{t})dz.$$

Therefore, in view of (4.18), it follows that

$$I_{3,1}(x_m,t_m)\to\tilde{\varphi}'(b)\cdot\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\alpha} e^{-z^2}dz, \quad I_{3,2}(x_m,t_m)\to 0 \quad \text{as} \;\; m\to\infty,$$

which yields

$$I_3(x_m,t_m)\to\tilde{\varphi}'(b)\cdot\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\alpha} e^{-z^2}dz \quad \text{as} \;\; m\to\infty. \tag{4.23}$$

Hence, (4.17) follows from (4.21)–(4.23) and (4.14).

A similar argument proves that

$$\tilde{u}_x(x,t)\to\tilde{\varphi}'(0) \quad \text{as} \quad (x,t)\to(0,0), \;\; (x,t)\in D_T^{\circ},$$

which completes the proof of (iii).

In order to complete the proof of Lemma 4.2 it remains to show that the condition (3.5) holds, i.e., $\tilde{u}(s(t),t)=0$ for $t\in[0,T)$. Since $\tilde{u}(x,t)$ satisfies (3.1)–(3.4) (as we proved above), by integrating the identity (4.5) over the domain $\varepsilon\le\tau\le t-\varepsilon, \quad \delta\le\xi\le s(\tau)-\delta, \quad \varepsilon,\delta>0$, and passing to limits, first as $\delta\to 0$ and then as $\varepsilon\to 0$, we obtain the integral representation (4.6). Now, in view of (4.8) and (4.13), it follows that

$$-\int_0^t N_\xi(x,t;s(\tau),\tau)g(\tau)d\tau+\int_0^t s'(\tau)N(x,t;s(\tau),\tau)g(\tau)d\tau=0,$$

where $g(t)=\tilde{u}(s(t),t),\ 0\le t<T$. Taking into account that $N_\xi=-G_x$ and passing to a limit as $x\to s(t)-0$, we obtain by Lemma 4.1

$$\frac{g(t)}{2}+\int_0^t G_x(s(t),t;s(\tau),\tau)g(\tau)d\tau+\int_0^t s'(\tau)N(s(t),t;s(\tau),\tau)g(\tau)d\tau=0. \tag{4.24}$$

We are going to explain that this integral equation for g has only the trivial solution $g(t)\equiv 0$. One can easily see that for any $T_1<T$ there is a constant $C>0$ such that for $\tau\in[0,T_1]$

$$|G_x(s(t),t;s(\tau),\tau)|\le\frac{C}{\sqrt{t-\tau}}, \quad |s'(\tau)N(s(t),t;s(\tau),\tau)|\le\frac{C}{\sqrt{t-\tau}}.$$

Now, by (4.24) it follows that

$$\sup_{[0,t_1]}|g(t)|\le 8C\sqrt{t_1}\sup_{[0,t_1]}|g(t)|, \quad t_1\in(0,T_1).$$

Choose t_1 so that $8C\sqrt{t_1} < 1$; then we have $g(t) = 0$ for $0 \le t \le t_1$.

The same argument shows that if $g(t) = 0$ for $t \in [0, t_0]$, then there is a $\delta > 0$ such that $g(t) = 0$ for $t \in [0, t_0 + \delta]$. Hence, $g(t) \equiv 0$ for $t \in [0, T)$, i.e., (3.5) holds. This completes the proof of Lemma 4.2. □

5. Local existence-uniqueness

We study the local existence–uniqueness properties of the system of nonlinear integral equations (4.11), (4.12) by employing the Banach Contraction Fixed Point Theorem.

Let $\varepsilon = \mathit{const} > 0$, and let E be the space of all pairs of continuous functions $(v(t), s'(t)),\ t \in [0, \varepsilon]$. Equipped with the norm

$$\|(v, s')\|_\varepsilon = \max\{\|v\|_\varepsilon, \|s'\|_\varepsilon\}, \quad \text{where} \quad \|v\|_\varepsilon = \sup_{[0,\varepsilon]} |v(t)|, \quad \|s'\|_\varepsilon = \sup_{[0,\varepsilon]} |s'(t)|,$$

E is a Banach space.

We fix a constant $T > 0$ and introduce the norms

$$\|f\|_T = \sup_{[0,T]} |f(t)|, \quad \|\tilde{f}\|_T = \sup_{[0,T]} |\tilde{f}(t)|, \quad \|\tilde{\varphi}'\|_b = \sup_{[0,b]} |\tilde{\varphi}'(x)|.$$

In the following we may assume that $\varepsilon < T$.

Consider in E the operator

$$\Phi(v, s') = (A(v, s'), B(v, s')), \tag{5.1}$$

where

$$A(v, s')(t) = 2A_1(v, s')(t) + 2A_2(v, s')(t) + 2A_3(v, s')(t) \quad \text{for} \quad t > 0 \tag{5.2}$$

with

$$A_1(v, s') = \int_0^t N_x(s(t), t; s(\tau), \tau) v(\tau) d\tau, \quad A_2(v, s') = -\int_0^t N_x(s(t), t; 0, \tau)\tilde{f}(\tau) d\tau,$$

$$A_3(v, s') = \int_0^b G(s(t), t; \xi, 0)\tilde{\varphi}'(\xi) d\xi,$$

and

$$B(v, s')(t) = \int_0^{s(t)} W(v, s')(x, t) dx, \tag{5.3}$$

with

$$W(v, s')(x, t) = f(t)e^{-\lambda t} - \tilde{\sigma} + e^{-\lambda t} \sum_{\nu=1}^{3} \int_0^x J_\nu(v, s')(\xi, t) d\xi, \tag{5.4}$$

where J_ν are the integrals introduced in (4.7).

In the above notations the system of integral equations (4.11) and (4.12) could be written as

$$(v, s') = \Phi(v, s'). \tag{5.5}$$

Next we prove that locally (say, for $0 < t \le \varepsilon$) the equation (5.5) has a unique solution by showing that for every large enough $M > 0$ there is an $\varepsilon > 0$ such

that the operator Φ maps the closed ball $\mathcal{B}_M = \{(v,s') : \ \|(v,s')\|_\varepsilon \le M\}$ into itself, and its restriction on $\mathcal{B}_M$ is a contraction mapping.

We impose the following a priori conditions on M and ε :

$$M > 1, \qquad \varepsilon < \min\{1, b/(2M)\}. \tag{5.6}$$

Then $\|s'\|_\varepsilon \le M$ implies $|s(t) - b| \le M\varepsilon < b/2$. Therefore,

$$\|s'\|_\varepsilon \le M \ \Rightarrow \ b/2 \le s(t) \le 3b/2 \quad \text{for} \quad 0 \le t \le \varepsilon. \tag{5.7}$$

Straightforward computations show that the operator Φ maps $\mathcal{B}_M$ into itself if

$$M \ge 1 + (8 + 5b^2)\|\tilde{\varphi}'\|_b + 4b^2\|f\|_T + 4b^2\tilde{\sigma} \tag{5.8}$$

and

$$\varepsilon \le \left[4(M + 4/b)^2 + (32/b)^2\|\tilde{f}\|_T + 5b^2M + 2\|\tilde{f}\|_T\right]^{-1}. \tag{5.9}$$

In order to prove that the operator $\Phi : \mathcal{B}_M \to \mathcal{B}_M$ is a contraction mapping we investigate the contraction properties of integral operators in (5.2)–(5.4). Since $A_1(v,s')$ and $A_2(v,s')$ are Voltera type integral operators, one can easily prove that there exists $\varepsilon > 0$ such that for $t \in [0,\varepsilon]$

$$|A_\nu(v_1,s_1')(t) - A_\nu(v_2,s_2')(t)| \le \frac{1}{4}\|(v_1 - v_2, s_1' - s_2')\|_\varepsilon, \quad \nu = 1,2, \tag{5.10}$$

whenever (v_1,s_1'), $(v_2,s_2') \in \mathcal{B}_M$.

Next we consider $A_3(v,s')$. Since $G(x,t;\xi,0) = K(x,t;\xi,0) - K(-x,t;\xi,0)$, we have $A_3(v_1,s_1')(t) - A_3(v_2,s_2')(t) = A_3^-(t) - A_3^+(t)$, where

$$A_3^\pm(t) = \frac{1}{2\sqrt{\pi t}}\int_0^b \left[e^{-\frac{(s_1(t)\pm\xi)^2}{4t}} - e^{-\frac{(s_2(t)\pm\xi)^2}{4t}}\right]\tilde{\varphi}'(\xi)d\xi.$$

By the elementary inequality $|e^{-x_1} - e^{-x_2}| \le |x_1 - x_2|e^{-\min(x_1,x_2)}$, $x_1, x_2 > 0$, the expression in the square brackets in A_3^+ does not exceed by absolute value

$$\frac{|s_1(t) - s_2(t)|}{4t}(s_1(t) + s_2(t) + 2\xi)\exp\left(-\frac{1}{4t}\min[(s_1(t) + \xi)^2, (s_2(t) + \xi)^2]\right)$$
$$\le \frac{5b}{4}\, e^{-b^2/16t} \cdot \|s_1' - s_2'\|_\varepsilon$$

because $b/2 \le s_i(t) \le 3b/2$, $i = 1,2$ (see (5.7)). Therefore, taking into account that $e^{-b^2/16t} < 16t/b^2$, we obtain

$$|A_3^+(t)| \le 10\sqrt{t}\|\tilde{\varphi}'\|_b\|s_1' - s_2'\|_\varepsilon \le \frac{1}{8}\|s_1' - s_2'\|_\varepsilon \quad \text{if} \quad t \le \varepsilon \le (80\|\tilde{\varphi}'\|_b)^{-2}.$$

In order to estimate A_3^- we write it in the form $A_3^-(t) = A_{3,1}^-(t) + A_{3,2}^-(t)$, where

$$A_{3,1}^-(t) = \frac{1}{2\sqrt{\pi t}}\int_0^{b-\delta} e^{-\frac{(s_1(t)-\xi)^2}{4t}}\left[1 - \exp\left(\frac{(s_1(t) - \xi)^2}{4t} - \frac{(s_2(t) - \xi)^2}{4t}\right)\right]\tilde{\varphi}'(\xi)d\xi,$$

$$A^-_{3,2}(t)=\frac{1}{2\sqrt{\pi t}}\int_{b-\delta}^{b}e^{-\frac{(s_1(t)-\xi)^2}{4t}}\left[1-\exp\left(\frac{(s_1(t)-\xi)^2}{4t}-\frac{(s_2(t)-\xi)^2}{4t}\right)\right]\tilde{\varphi}'(\xi)d\xi.$$

and $\delta=2Mt^{1/4}$.

From (5.7) it follows that $|s_1(t)+s_2(t)-2\xi|\le 3b$ which implies

$$\left|\frac{(s_1(t)-\xi)^2}{4t}-\frac{(s_2(t)-\xi)^2}{4t}\right|=\frac{|s_1(t)-s_2(t)|}{4t}|s_1(t)+s_2(t)-2\xi|\le\frac{3b}{4}\cdot\|s_1'-s_2'\|_\varepsilon.$$

Therefore, by the inequality $|e^x-1|\le|x|e^{|x|}$, the expression in the square brackets in $A^-_{3,1}(t)$ does not exceed by absolute value $(3b/4)\exp(3bM/2)\|s_1'-s_2'\|_\varepsilon$. Thus, it follows

$$\left|A^-_{3,1}(t)\right|\le\frac{3b}{4}\,e^{3bM/2}\,\|\tilde{\varphi}'\|_b\cdot\|s_1'-s_2'\|_\varepsilon\cdot\int_0^{b-\delta}\frac{1}{2\sqrt{t}}\,e^{-\frac{(s_1(t)-\xi)^2}{4t}}d\xi.$$

Performing the change of variable $z=\frac{\xi-s_1(t)}{2\sqrt{t}}$ in the latter integral, and estimating from above the resulting integral, we obtain

$$\int_{-\frac{s_1(t)}{2\sqrt{t}}}^{\frac{b-\delta-s_1(t)}{2\sqrt{t}}}e^{-z^2}dz\le\frac{b-\delta}{2\sqrt{t}}e^{-\frac{(b-\delta-s_1(t))^2}{4t}}\le\frac{b}{2\sqrt{t}}e^{-\frac{M^2}{4\sqrt{t}}}\le 8b\cdot t^{1/2}$$

because

$$\delta+(s_1(t)-b)\ge 2Mt^{1/4}-\|s_1'\|_\varepsilon\cdot t\ge 2Mt^{1/4}-Mt\ge Mt^{1/4},$$

and (by (4.4) and (5.6)) $t^{-1}\exp(-M^2/4\sqrt{t})\le 16$. Therefore,

$$\left|A^-_{3,1}(t)\right|\le 6b^2\exp(3bM/2)\|\tilde{\varphi}'\|_b\cdot\|s_1'-s_2'\|_\varepsilon\cdot\sqrt{t}.$$

Next we estimate $A^-_{3,2}(t)$. If $\xi\in[b-\delta,b]$, then (since $s_1(0)=s_2(0)=b$)

$$\begin{aligned}|s_1(t)+s_2(t)-2\xi|&=|s_1(t)-s_1(0)+s_2(t)-s_2(0)+2(b-\xi)|\\&\le(\|s_1'\|_\varepsilon+\|s_2'\|_\varepsilon)\cdot t+2\delta\le 2Mt+4Mt^{1/4}\le 6Mt^{1/4},\end{aligned}$$

which implies

$$\left|\frac{(s_1(t)-\xi)^2}{4t}-\frac{(s_2(t)-\xi)^2}{4t}\right|=\frac{|s_1(t)-s_2(t)|}{4t}|s_1(t)+s_2(t)-2\xi|\le\frac{3}{2}Mt^{1/4}\|s_1'-s_2'\|_\varepsilon.$$

Estimating the expression in the square brackets in $A^-_{3,2}(t)$ as in the case of $A^-_{3,1}(t)$ and taking into account (4.3), we obtain

$$\left|A^-_{3,2}(t)\right|\le\frac{3}{2}Mt^{1/4}\exp(3M^2)\|\tilde{\varphi}'\|_b\|s_1'-s_2'\|_\varepsilon.$$

Thus,

$$|A^-_3(t)|\le\left[6b^2\exp(3bM/2)+3M\exp(3M^2)\right]\|\tilde{\varphi}'\|_b\|s_1'-s_2'\|_\varepsilon\cdot t^{1/4},$$

which implies that

$$|A^-_3(t)|\le\frac{1}{8}\|s_1'-s_2'\|_\varepsilon$$

if $$t\le\varepsilon\le 8^{-4}\left[6b^2\exp(3bM/2)+3M\exp(3M^2)\right]^{-4}\|\tilde{\varphi}'\|_b^{-4}.$$

Now, the estimates for $A_3^-(t)$ and $A_3^+(t)$ imply that if

(5.11)
$$\varepsilon \le \min\left\{80\|\tilde{\varphi}'\|_b)^{-2},\, 8^{-4}\left[6b^2\exp(3bM/2)+3M\exp(3M^2)\right]^{-4}\|\tilde{\varphi}'\|_b^{-4}\right\},$$

then

$$|A_3(v_1,s_1')(t)-A_3(v_2,s_2')(t)| \le \frac{1}{4}\|s_1'-s_2'\|_\varepsilon. \tag{5.12}$$

The estimates (5.10) hold if ε satisfies inequalities similar to (5.9) and (5.11). Moreover, one can prove that the operator B is a contraction in $\mathcal{B}_M$ if ε satisfies similar restrictions. We omit the details, but it is important to note that the right-hand sides of (5.9), (5.11) and the analogous inequalities (which guarantee that the operator Φ is a contraction on $\mathcal{B}_M$ with a contraction coefficient < 1) are given by expressions that decrease if the parameters involved (such as $b, M, \|\tilde{\varphi}'\|_b, \|f\|_T, \|\tilde{f}\|_T$) increase.

Therefore, applying the Banach Contraction Fixed Point Theorem, we obtain the following statement.

Lemma 5.1. *(a) For each constant $M > 1$ which satisfies (5.8) there is a constant $\varepsilon > 0$ such that the system of integral equations (4.11) and (4.12) (with $s(t) = b + \int_0^t s'(\tau)d\tau$) has a unique solution $(v(t),\, s'(t))$, $t \in [0,\varepsilon]$, such that $\|v\|_\varepsilon \le M$ and $\|s'\|_\varepsilon \le M$.*

(b) The constant ε may be chosen so that

$$\varepsilon = h(M, b, 1/b, \|f\|_T, \|\tilde{f}\|_T, \|\tilde{\varphi}'\|_b), \tag{5.13}$$

where $h(y_1,y_2,y_3,y_4,y_5,y_6)$, $y_i > 0$, is a monotone decreasing function with respect to each argument y_i, $i = 1,\dots,6$.

Next we prove uniqueness of solutions of Problem $\tilde{P}$.

Lemma 5.2. *For each $T \le \infty$, Problem $\tilde{P}$ has at most one solution for $t \in [0,T)$.*

Proof. Suppose that $(\tilde{u}_1(x,t), s_1(t))$ and $(\tilde{u}_2(x,t), s_2(t))$ are two solutions of Problem $\tilde{P}$ on the interval $[0,T)$, $T \le \infty$. Then, in view of Lemma 4.2, the pairs of functions $(v_1(t), s_1'(t))$ and $(v_2(t), s_2'(t))$, where $v_1(t) = \frac{\partial \tilde{u}_1}{\partial x}(s_1(t),t)$, $v_2(t) = \frac{\partial \tilde{u}_2}{\partial x}(s_2(t),t)$, are solutions of the system of integral equations (4.11), (4.12). Fix $\varepsilon_0 < T$ and choose $M > 1$ so that

$$M \ge \max\{\|v_1(t)\|_{\varepsilon_0}, \|s_1'\|_{\varepsilon_0}, \|v_2(t)\|_{\varepsilon_0}, \|s_2'\|_{\varepsilon_0}\}.$$

By Lemma 5.1, there is a positive constant $\varepsilon < \varepsilon_0$ such that the pairs $(v_1(t), s_1'(t))$ and $(v_2(t), s_2'(t))$ coincide on the interval $[0,\varepsilon]$. Therefore, the integral representation (4.10) implies

$$\tilde{u}_1(x,t) = \tilde{u}_2(x,t), \quad s_1(t) = s_2(t) \quad \text{if} \quad 0 \le t \le \varepsilon,\ 0 \le x \le s(t).$$

Having proved uniqueness for a small time interval $0 \le t \le \varepsilon$, we can proceed in a similar way, step by step, to get uniqueness for all $t > 0$. Let

$t_0 < T$ be a positive number such that

$$\tilde{u}_1(x,t) = \tilde{u}_2(x,t), \quad s_1(t) = s_2(t) \quad \text{if} \quad 0 \le t \le t_0,\ 0 \le x \le s(t). \tag{5.14}$$

Then $(e^{-\lambda t_0}\tilde{u}_1(x, t+t_0), s_1(t+t_0))$ and $(e^{-\lambda t_0}\tilde{u}_2(x, t+t_0), s_2(t+t_0))$ are two solutions of Problem $\tilde{P}$ on the interval $[0, T-t_0)$, if considered with $f_1(t) = f(t+t_0)$,

$$\tilde{f}_1(t) = e^{-\lambda t_0}\tilde{f}(t+t_0),\ \tilde{\varphi}_1(x) = e^{-\lambda t_0}\tilde{u}_1(x,t_0) = e^{-\lambda t_0}\tilde{u}_2(x,t_0),\ b_1 = s_1(t_0) = s_2(t_0)$$

instead of $f(t), \tilde{f}(t), \tilde{\varphi}(x)$ and b. By the above argument, there is a constant $\varepsilon_1 > 0$ such that

$$\tilde{u}_1(x,t) = \tilde{u}_2(x,t), \quad s_1(t) = s_2(t) \quad \text{if} \quad 0 \le t \le t_0 + \varepsilon_1,\ 0 \le x \le s(t).$$

Therefore, (5.14) holds for each $t_0 < T$, i.e. the solutions $(\tilde{u}_1(x,t), s_1(t))$ and $(\tilde{u}_2(x,t), s_2(t))$ coincide on $[0, T)$. □

6. Existence of global solution

Lemma 4.2 and Lemma 5.1 guarantee that the Problem $\tilde{P}$ has a solution for $0 \le t < \varepsilon$ for sufficiently small $\varepsilon > 0$. In order to prove the existence of a global solution we need a priori estimates for $s(t)$ and $\tilde{u}_x(x,t)$.

By Lemma 2.2, there are constants $C_1 = C_1(T)$ and $C_2 = C_2(T)$ such that

$$|s'(t)| \le C_1, \quad 1/C_2 \le |s(t)| \le C_2, \quad 0 \le t < T. \tag{6.1}$$

Lemma 6.1. *Suppose that the pair of functions $(\tilde{u}(x,t), s(t))$ is a solution of Problem $\tilde{P}$ for $t \in [0,T)$, $0 < T < \infty$. Then*

$$\Psi_T := \sup_{D_T} |\tilde{u}_x(x,t)| < \infty, \tag{6.2}$$

where $D_T = \{(x,t) :\ 0 \le x \le s(t),\ 0 \le t < T\}$.

Proof. It is enough to prove that

$$m := \sup_{[0,T)} |v(t)| < \infty, \tag{6.3}$$

where $v(t) = \tilde{u}_x(s(t), t)$. Indeed, by (4.15), $\tilde{u}_x(x,t) = \sum_{\nu=1}^{3} I_\nu(x,t)$, where the integrals $I_\nu(x,t)$ are given by (4.16).

We have $I_1(x,t) = I_{1,1}(x,t) + I_{1,2}(x,t)$, where $I_{1,1}(x,t)$ and $I_{1,2}(x,t)$ are given in (4.19) and (4.20). First we estimate $|I_{1,1}(x,t)|$:

$$|I_{1,1}(x,t)| \le m \cdot \left(\frac{1}{\sqrt{\pi}} \int_0^t \frac{|s(t)-s(\tau)|}{4(t-\tau)^{3/2}} e^{-\frac{(s(\tau)-x)^2}{4(t-\tau)}} d\tau \right.$$

$$\left. + \frac{1}{\sqrt{\pi}} \int_0^t \frac{|s(t)-x|}{4(t-\tau)^{3/2}} e^{-\frac{(s(t)-x)^2}{4(t-\tau)}} \exp\left[\frac{(s(t)-x)^2 - (s(\tau)-x)^2}{4(t-\tau)} \right] d\tau \right).$$

From (6.1) it follows that $|s(t) - s(\tau)|/(t-\tau) \le C_1$, so the first integral in the brackets does not exceed $\int_0^t C_1/(2\sqrt{t-\tau}) d\tau \le C_1\sqrt{T}$. By (6.1), the

expression in the square brackets in the integrand of the second integral can be estimated from above by

$$\frac{|s(t)-s(\tau)|}{4(t-\tau)}|s(t)+s(\tau)-2x| \le C_1C_2.$$

Therefore, in view of (4.2), the second integral does not exceed $e^{C_1C_2}$. Hence,

$$|I_{1,1}(x,t)| \le m\cdot\left(C_1\sqrt{T}+e^{C_1C_2}\right).$$

By (6.1) we have $C_2^{-1} \le x+s(\tau) \le 2C_2$ for $x<s(t)$. From these inequalities and (4.2) it follows

$$|I_{1,2}(x,t)| \le m\cdot 2C_2^2\int_0^t K_x(C_2^{-1},t;0,\tau)d\tau \le m\cdot 2C_2^2.$$

On the other hand (4.2) and (4.3) imply

$$|I_2(x,t)| \le \|\tilde f\|_T, \quad |I_3(x,t)| \le 2\|\tilde\varphi'\|_b. \tag{6.4}$$

Hence,

$$\sup_{D_T}|\tilde u_x(x,t)| \le m\cdot\left(C_1\sqrt{T}+e^{C_1C_2}+2C_2^2\right)+\|\tilde f\|_T+2\|\tilde\varphi'\|_b,$$

i.e., (6.3) implies (6.2).

Next we prove (6.3). By (4.11), $v(t)=2\sum_{\nu=1}^3 I_\nu(s(t),t)$, where the integrals $I_\nu(x,t)$ are given by (4.16).

First we consider $I_1(s(t),t)=\int_0^t N_x(s(t),t;s(\tau),\tau)v(\tau)d\tau$. Since $N(x,t;\xi,\tau)=K(x,t;\xi,\tau)+K(-x,t;\xi,\tau)$, we have

$$|N_x(s(t),t;s(\tau),\tau)| \le \frac{|s(t)-s(\tau)|}{4(t-\tau)^{3/2}}e^{-\frac{(s(t)-s(\tau))^2}{4(t-\tau)}}+\frac{|s(t)+s(\tau)|}{4(t-\tau)^{3/2}}e^{-\frac{(s(t)+s(\tau))^2}{4(t-\tau)}}.$$

From (6.1) it follows that $|s(t)-s(\tau)|/(t-\tau)\le C_1$, so the first term on the right in the above inequality is less than $C_1/(4\sqrt{t-\tau})$. On the other hand, (6.1) implies $2/C_2 \le s(t)+s(\tau) \le 2C_2$. Therefore, in view of (4.4),

$$\frac{|s(t)+s(\tau)|}{4(t-\tau)^{3/2}}e^{-\frac{(s(t)+s(\tau))^2}{4(t-\tau)}} \le \frac{2C_2}{4(t-\tau)^{3/2}}e^{-\frac{1}{C_2^2(t-\tau)}} \le C_2^3\frac{1}{2\sqrt{t-\tau}}.$$

Thus, we obtain

$$|N_x(s(t),t;s(\tau),\tau)| \le (C_1/2+C_2^3)\frac{1}{2\sqrt{t-\tau}}, \quad 0\le\tau<t<T. \tag{6.5}$$

Let $\delta\in(0,T)$ (later we will choose δ sufficiently small), and let

$$\mu(t)=\sup\{|v(\tau)|:\ 0\le\tau\le t\}.$$

For $t\in(T-\delta,T)$, (6.5) implies

$$\begin{aligned}|I_1(s(t),t)| &\le \mu(T-\delta)\int_0^{T-\delta}|N_x(s(t),t;s(\tau),\tau)|d\tau+\mu(t)\int_{T-\delta}^t|N_x(s(t),t;s(\tau),\tau)|d\tau\\ &\le \mu(T-\delta)\cdot(C_1/2+C_2^3)\sqrt{T}+\mu(t)\cdot(C_1/2+C_2^3)\sqrt{\delta}.\end{aligned}$$

Therefore, in view of (4.11) and (6.4), we obtain

$$|v(t)| \le \mu(T-\delta)\cdot(C_1+2C_2^3)\sqrt{T}+\mu(t)\cdot(C_1+2C_2^3)\sqrt{\delta}+2\|\tilde f\|_T+4\|\tilde\varphi'\|_b \tag{6.6}$$

for $t \in (T-\delta, T)$.

Choose δ so that

$$(C_1 + 2C_2^3)\sqrt{\delta} < 1/2.$$

Then (6.6) implies

$$\mu(t) \le 2\mu(T-\delta)(C_1 + 2C_2^3)\sqrt{T} + 4\|\tilde f\|_T + 8\|\tilde\varphi'\|_b \quad \text{for } t \in (T-\delta, T).$$

Hence, $m = \sup\{\mu(t),\ t \in [0,T)\} < \infty$, i.e., (6.3) holds, which completes the proof of Lemma 6.1. □

Proof of Theorem 3.2. By Lemma 5.2, Problem $\tilde P$ has at most one global solution. Now we prove that Problem $\tilde P$ has a global solution. Assume the contrary, and let T be the greatest positive number such that Problem $\tilde P$ has a solution for $t \in [0,T)$. Let the pair of functions $\tilde u(x,t)$ and $s(t)$ be a solution of Problem $\tilde P$ for $t \in [0,T)$.

For each $t_0 < T$ we can consider a "modified" Problem $\tilde P$ with data

$$f_1(t) = f(t_0+t),\ \ \tilde f_1(t) = e^{-\lambda t_0}\tilde f(t_0+t),\ \ \tilde\varphi_1(x) = e^{-\lambda t_0}\tilde u(x,t_0),\ \ b_1 = s(t_0) \tag{6.7}$$

instead of $f(t), \tilde f(t), \tilde\varphi(x)$ and b. By the local existence–uniqueness result given in Lemma 5.1, for each $M_1 > 1$ which satisfies

$$M_1 \ge 1 + (8+5b_1^2)\|\tilde\varphi_1'\|_{b_1} + 4b_1^2\|f_1\|_T + 4b_1^2\tilde\sigma \tag{6.8}$$

and each $\varepsilon > 0$ with

$$\varepsilon \le h(M_1, b_1, 1/b_1, \|f_1\|_T, \|\tilde f_1\|_T, \|\tilde\varphi_1'\|_{b_1}), \tag{6.9}$$

the "modified" Problem $\tilde P$ has a solution $(\tilde u_1(x,t), s_1(t))$ for $0 \le t < \varepsilon$. Then, the pair $(\tilde U(x,t), S(t))$ with

$$\tilde U(x,t) = \begin{cases} \tilde u(x,t) & 0 \le t \le t_0 \\ e^{\lambda t_0}\tilde u_1(x, t-t_0) & t_0 \le t < t_0+\varepsilon \end{cases}, \quad S(t) = \begin{cases} s(t) & 0 \le t \le t_0 \\ s_1(t-t_0) & t_0 \le t < t_0+\varepsilon \end{cases}$$

is a solution of Problem $\tilde P$ for $0 \le t < t_0 + \varepsilon$.

Moreover, in view of the a priori estimates given in Lemma 2.2 and Lemma 6.1, by Lemma 5.1 we can choose *one and the same* ε for every $t_0 < T$. Indeed, let us set

$$\tilde M = 1+(8+5C_2^2)\Psi_T+4C_2^2\|f\|_{2T}+4C_2^2\tilde\sigma, \quad \tilde\varepsilon = h(\tilde M, C_2, C_2, \|f\|_{2T}, \|\tilde f\|_{2T}, \Psi_T).$$

Therefore, choosing $t_0 > T - \tilde\varepsilon$, we get the existence of a solution of Problem $\tilde P$ for $t \in [0, t_0+\tilde\varepsilon)$ with $t_0 + \tilde\varepsilon > T$, which contradicts the choice of T. Hence Problem $\tilde P$ has a global solution for $t \in [0,\infty)$, i.e., Theorem 3.2 holds.

In view of Lemma 3.1, this implies that Problem P has a unique global solution, i.e., Theorem 1.1 holds as well.

7. Concluding remarks

1. During the last 40 years various mathematical models for evolution of tumors have been developed and analyzed – see the survey papers [1, 9] and the bibliography therein. Some of those models are in the form of free boundary problems for partial differential equations, whereby the tumor surface is a free boundary and the tumor growth is determined by the level of a diffusing nutrient concentration [10, 11] (see also [1]–[6], [8]). The main physical and biological concepts underlying such type of models are the mass conservation law and reaction–diffusion processes within the tumor. Usually additional geometric assumptions on the shape of the tumor are imposed – see, for instance, [10, 8], where the tumor is supposed to be spherically symmetric.

A slight modification of those models is considered in [14]. It describes the growth of an avascular solid tumor which receives nutrient supply via a diffusion process only through some part of its boundary (called *base* of the tumor), and it is assumed that there is no nutrient flow through the remaining part of the boundary. Moreover, the tumor is supposed to be thin and approximately disc-shaped, so only one spatial dimension, say x, is considered. With tumor's base situated at $x = 0$ the nutrient concentration $\sigma(x, t)$ satisfies the reaction–diffusion equation

$$c\frac{\partial \sigma}{\partial t} = \frac{\partial^2 \sigma}{\partial x^2} - \lambda\sigma, \quad 0 < x < s(t), \;\; t > 0, \tag{7.1}$$

where $s(t) > 0$ is the tumor's thickness at time t, $\lambda = const > 0$, $\lambda\sigma$ is the nutrient consumption rate, and $c > 0$ is a dimensionless constant coming as a ratio of the nutrient diffusion time scale to the tumor growth time scale.

Following [10], it is assumed that all tumor cells are physically identical in volume and mass, and that the cell density is constant throughout the tumor. As in [8], the cell proliferation rate within the tumor is given by $P(\sigma) = \mu(\sigma - \tilde{\sigma})$, where μ and $\tilde{\sigma}$ are positive constants. These assumptions lead to the equation

$$s'(t) = \mu \int_0^{s(t)} (\sigma(x, t) - \tilde{\sigma}) dx, \quad t \geq 0. \tag{7.2}$$

In addition, the following *mixed type* boundary conditions hold:

$$\sigma(0, t) = f(t), \qquad t \geq 0, \tag{7.3}$$

$$\sigma(x, 0) = \varphi(x), \quad x \in [0, b], \;\; s(0) = b > 0, \quad \varphi(0) = f(0), \tag{7.4}$$

$$\frac{\partial \sigma}{\partial x}(s(t), t) = 0, \quad t > 0, \tag{7.5}$$

where $f(t) > 0$ is the external nutrient concentration at the base of the tumor at time t, $\varphi(x) > 0$ is the initial nutrient concentration within the tumor, and the condition (7.5) comes because it is assumed that there is no nutrient transfer through the free boundary $x = s(t)$.

The parameters c, μ and λ in (7.1)–(7.5) depend on the choice of time and length units. One may scale out x and t in an appropriate way in order to get $c = 1$ and $\mu = 1$ (λ may change as well), which shows that Problem (7.1)–(7.5) is equivalent to Problem P.

2. Another interesting question in the study of mathematical models of tumor growth is under what conditions does the tumor grow, shrink or become dormant. In order to answer that question one needs to find the stationary solution (which gives the dormant case) and analyze its asymptotic stability (see [8, 5, 6] and the bibliography there).

In the case of Problem P, if $f(t) = \bar{\sigma} = const$ then it is easy to see that the stationary solution is given by the pair $(\bar{u}(x), \bar{b})$, where

$$\bar{u}(x) = \bar{\sigma}\,\frac{\cosh\left(\sqrt{\lambda}(x-\bar{b})\right)}{\cosh\left(\sqrt{\lambda}\,\bar{b}\right)} \tag{7.6}$$

and $\bar{b}$ is determined by the equation

$$\bar{\sigma}\,\tanh\left(\bar{b}\sqrt{\lambda}\right) = \tilde{\sigma}\bar{b}\,\sqrt{\lambda}\,. \tag{7.7}$$

3. In (1.6) of Theorem 1.1, the assumptions $f(t) > 0$ for $t \in [0, \infty)$ and $\varphi(x) > 0$ for $x \in [0, b]$ come from the corresponding mathematical model (Section 7.1). However, the result stated in Theorem 1.1 remains valid without those requirements.

Indeed, let $f(t) \in C^1([0, \infty))$ and $\varphi(x) \in C^2([0, b])$ be arbitrary functions. Then, under the assumptions of Lemma 2.2, the following a priori estimates hold:

$$|u(x,t)| \le C_T, \quad 0 \le x \le s(t), \;\; 0 \le t < T, \tag{7.8}$$

(7.9)
$$-(C_T + \tilde{\sigma})s(t) \le s'(t) \le (C_T - \tilde{\sigma})s(t), \quad be^{-(C_T+\tilde{\sigma})t} \le s(t) \le be^{(C_T-\tilde{\sigma})t},$$

where $b = s(0)$, $C_T = \max\left(\sup_{[0,T)} |f(t)|,\ \sup_{[0,b]} |\varphi(x)|\right)$. The proof of Theorem 1.1 is the same, but one needs to use the estimates (7.8) and (7.9) instead of (2.3) and (2.4) in Lemma 2.2.

4. It is known that the free boundary in the one-dimensional Stefan problem (see [7, Ch. 8], [12], [2, Ch. 17]) is a C^∞-curve (see [3, 4, 13] and the bibliography therein). In the context of tumor models a similar result is proven in [6, Theorem 4.1].

In the case of our Problem P it is easy to see that $s(t) \in C^2([0, \infty))$. Indeed, since $u_t(x,t)$ is defined and continuous for $0 \le x \le s(t)$, $t > 0$, from (1.2) it follows

$$s''(t) = [u(s(t),t) - \tilde{\sigma}]s'(t) + \int_0^{s(t)} u_t(x,t)dx.$$

Therefore, taking into account that

$$\int_0^{s(t)} u_t(x,t)dx = \int_0^{s(t)} (u_{xx} - \lambda u)dx = u_x(s(t),t) - u_x(0,t) - \lambda \int_0^{s(t)} u(x,t)dx$$

we obtain, in view of (1.2) and (1.5),

$$s''(t) = [u(s(t),t) - \tilde{\sigma} - \lambda]s'(t) - u_x(0,t) - \lambda\,\tilde{\sigma} s(t),$$

where the expression on the right is a continuous function for $t \geq 0$, i.e., $s(t) \in C^2([0,\infty))$.

However, higher derivatives of $s(t)$ may not exist if we assume $f \in C^1([0,\infty))$ only, since the condition (1.2) in Problem P is nonlocal (compare with the case of one-dimensional Stefan problem, where the infinite differentiability of the free boundary does not require infinite differentiability of the boundary data at $x = 0$ – see [13]). In our case, one can prove the following: *In Problem P, the free boundary* $x = s(t)$, $t \in (0,\infty)$ *is an infinitely differentiable curve if and only if* $f(t) \in C^\infty((0,\infty))$. We will present the details somewhere else.

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